Stability boundary analysis of nonlinear dynamics subject to state limits

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Abstract: In the spirit of Morse-Smale systems, the paper analyzes the structure of stability boundary of a stable equilibrium point for nonlinear dynamics subject to state limits. Presence of state limits implies that the underlying dynamics does not satisfy the Lipschitz condition for solution existence/uniqueness. There does not exist a smooth flow for the dynamics thus complicating traditional analysis of stability boundary. By analyzing geometric properties of the solutions of the constrained dynamics, the paper establishes a characterization of the stability boundary under rather strong assumptions as a first step towards detailed boundary characterization.

Key words: Nonlinear dynamics, state limits, state saturation, stability analysis.

I. INTRODUCTION

Consider the class of nonlinear dynamics
\[ \Sigma : \frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n \] (1)
where the states x are limited to
\[ X = \{ x \in \mathbb{R}^n, x_{li} \leq x \leq x_{ui} \text{ for } i=1,2,\ldots,n \} \] (2)
The lower and upper limits x_{li} and x_{ui} are assumed to be $-\infty$ and $+\infty$, if state x_{i} is not limited in the respective direction. We will assume that the function f : $\mathbb{R}^n \rightarrow \mathbb{R}^n$ is smooth in the neighborhood of the constrained state space X. Given the constrained dynamics (1) and (2), the first problem is to generate the modified “vector field” say $\bar{f}(x)$ that renders the dynamics (1) consistent with the constrained formulation (2). In engineering sense, we say that the state x_{i} gets stuck at its limit say x_{ui} whenever the state x_{i} is pushed above its limit by the derivative $\frac{dx_{i}}{dt}=f_{i}(x) (>0)$. The formulation of the modified vector field $\bar{f}(x)$ can thus be summarized in an engineering sense as
\[
\begin{align*}
\bar{f}_{i}(x) &= 0 \quad \text{if } x_{i} = x_{ui} \text{ and } f_{i}(x) \geq 0 \\
&= 0 \quad \text{if } x_{i} = x_{li} \text{ and } f_{i}(x) \leq 0 \\
&= f_{i}(x) \quad \text{otherwise}
\end{align*}
\] (3)

Hardlimits on state x_{i} are commonly known as “nonwindup limits” in the power engineering literature [1], while the name “state limits” is more common in the control literature and in circuits literature. Equation (3) implies that when a trajectory reaches a state limit surface say $X_{ui} = \{ x_{i} = x_{ui} \}$, and if the state x_{i} tries to move past the limit with $f_{i} > 0$, then the state x_{i} gets stuck at the limit x_{ui}, and therefore, $\bar{f}_{i}$ is annihilated to be zero. At these points then, the vector field $\bar{f}$ will be discontinuous and specifically, is not locally Lipschitz. Therefore, it is not at all clear when solutions exist for the dynamics (1) and whether they are unique. This problem was analyzed in depth in [2] and we summarize the existence result first.

In order to construct the solutions of the dynamics (1)-(2) from the engineering definition (3), we need to make the following technical assumption [2]. Suppose $x(t_{0})=p$, and say $p_{i} = x_{ui}$ so that p belongs to the state limit set $\{ x_{i} = x_{ui} \}$. Then we say that the local solution say $x(t)$ of (1)-(2) with $x(t_{0})=p$ has order of contact k with $\{ x_{i} = x_{ui} \}$ at p if all the first k-1 derivatives of the x_{i} coordinate of x(t) vanish at p, and the k-th derivative is nonzero.

(SL0) All non-constant solution curves of the dynamics (1)-(2) have finite order of contact with all the state limit surfaces, the boundary of X.

Details on assumption (SL0) and the subsequent mathematical construction of the solution sets for (1)-(2) can be seen in [2]. The following theorem from [2] summarizes the solution properties.

Theorem 1 [2]. Under Assumption (SL0), there exists a unique vector field $\bar{f}$ which is compatible with the state limits in the sense of dynamics (1)-(2). The vector field is piecewise smooth in the sense that there exists a decomposition of X into embedded submanifolds with the property that the restriction of $\bar{f}$ onto these submanifolds are smooth vector fields. For every point $p \in X$, there exists a unique solution $x(t)$ with $x(0)=p$ forward in time that satisfies (1)-(2) with a maximal interval of existence $[0,T(p))$. The vector field $\bar{f}$ generates a positive semi-flow.

For the state limited dynamics (1)-(2), unique solutions exist for any initial condition in the constrained state space X in forward time, whereas neither existence nor uniqueness holds for solutions in negative time.